AP Calculus BC

Unit 11 - Power Series

AP Calculus BC – Worksheet 87

Introduction to Power Series; Interval and Radius of Convergence

Find the series' radius and interval of convergence.

1) $\sum_{n=0}^{\infty} x^n$	2) $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$	3) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$
4) $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$	5) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$	6) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$
7) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$	8) $\sum_{n=0}^{\infty} \frac{\sqrt{n} x^n}{3^n}$	9) $\sum_{n=0}^{\infty} n!(x-4)^n$

AP Multiple Choice

What is the value of $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{5^n}$?

(A) $-\frac{15}{8}$ (B) $-\frac{9}{8}$ (C) $-\frac{3}{8}$ (D) $\frac{9}{8}$ (E) $\frac{15}{8}$

What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(x + \frac{3}{2}\right)^n$ converges?

(A) $-\frac{5}{2} < x < -\frac{1}{2}$ (B) $-\frac{5}{2} < x \le -\frac{1}{2}$ (C) $-\frac{5}{2} \le x < -\frac{1}{2}$

(D) $-\frac{1}{2} < x < \frac{1}{2}$ (E) $x \le -\frac{1}{2}$

Answers to Worksheet 87

1) $R=1;\ -1<x<1$	2) $R=\frac{1}{4};\ -\frac{1}{2}<x<0$	3) $R=\frac{1}{3};\ \frac{1}{3}\le x<1$
4) $R=1;\ -3<x\le -1$	5) $R=1;\ 0\le x\le 2$	6) ∞; convergent for all x
7) ∞; convergent for all x	8) $R=3;\ -3<x<3$	9) $R=0$; convergent for $x=4$ only

AP Calculus BC – Worksheet 88 **Power Series**

Find the series' radius and interval of convergence.

1) $\sum_{n=0}^{\infty}(x+5)^n$	2) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$	3) $\sum_{n=1}^{\infty}\frac{nx^n}{n+2}$
4) $\sum_{n=1}^{\infty}\frac{3^n x^n}{n!}$	5) $\sum_{n=1}^{\infty}\frac{n(x+3)^n}{5^n}$	6) $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x+2)^n}{n2^n}$

7) For what values of x does the series $1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots$ converge?

For #8-10, write out the first six terms of each series. Determine the radius and interval of convergence for each.

8) $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n+1)!}$	9) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$	10) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

AP Multiple Choice

The radius of convergence for the power series $\sum_{n=1}^{\infty}\frac{(x-3)^{2n}}{n}$ is equal to 1. What is the interval of convergence?

(A) $-4 \le x < -2$ (B) $-1 < x < 1$ (C) $-1 \le x < 1$

(D) $2 < x < 4$ (E) $2 \le x < 4$

What is the radius of convergence of the series $\sum_{n=0}^{\infty}\frac{(x-4)^{2n}}{3^n}$?

(A) $2\sqrt{3}$ (B) 3 (C) $\sqrt{3}$ (D) $\frac{\sqrt{3}}{2}$ (E) 0

AP Calculus BC – Worksheet 89 **Taylor/Maclaurin Polynomials**

Find the third-degree Taylor polynomial generated by f centered at $x = a$.

1) $f(x) = \ln x$; $a = 1$	2) $f(x) = \sin x$; $a = \frac{\pi}{4}$
3) $f(x) = \sqrt{x}$; $a = 4$	4) $f(x) = e^x$, $a = 2$

Find the first four terms and the general term for the Maclaurin series of each function.

5) $f(x) = e^{-x}$	6) $f(x) = \frac{1}{1+x}$
7) $f(x) = \sin 3x$	8) $f(x) = (x+1)^3$

AP Multiple Choice

Let $P(x) = 3 - 3x^2 + 6x^4$ be the fourth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f^{(4)}(0)$?

(A) 0 (B) $\frac{1}{4}$ (C) 6 (D) 24 (E) 144

ANSWERS

1) $T(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3}(x-1)^3$	2) $T(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$
3) $T(x) = 2 + \frac{1}{2^2}(x-4) - \frac{1}{2!\cdot 2^5}(x-4)^2 + \frac{1}{2!\cdot 2^8}(x-4)^3$	4) $T(x) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3$
5) $M(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^n}{n!}x^n$	6) $M(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$
7) $M(x) = 3x - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7 + \cdots + \frac{3^{2n+1}}{(2n+1)!}x^{2n+1}$	8) $M(x) = 1 + 3x + 3x^2 + x^3$

1. e^{-2x} (Find the 4th degree Maclaurin polynomial.)
2. $\sin 2x$ (Find the 3rd degree Maclaurin polynomial.)
3. $\tan x$ (Find the 3rd degree Maclaurin polynomial.)

Find the 3rd degree Taylor polynomial about x = a for the following functions:

4. e^x $a = 1$
5. $\ln x$ $a = 1$
6. $\sqrt{x}$ $a = 4$

Find the Maclaurin series for the given function. Express your answer in sigma notation.

7. e^{-x}
8. $\dfrac{1}{1+x}$

Find the Taylor series about x = a and express your answer in sigma notation.

9. $\dfrac{1}{x}$ $a = -1$
10. e^x $a = 2$
11. $\ln x$ $a = 1$

Answers:

1. $1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4$
2. $2x - \frac{4}{3}x^3$
3. $x + \frac{1}{3}x^3$
4. $e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3$
5. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$
6. $2 + \frac{1}{4}(x-4) - \frac{1}{2^6}(x-4)^2 + \frac{1}{2^9}(x-4)^3$
7. $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$
8. $1 - x + x^2 - x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n$
9. $-1 - (x+1) - (x+1)^2 - (x+1)^3 - \ldots = \sum_{n=0}^{\infty} -(x+1)^n$
10. $e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \ldots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$
11. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$

AP Multiple Choice

The Taylor polynomial of degree 100 for the function f about $x = 3$ is given by

$$P(x) = (x-3)^2 - \frac{(x-3)^4}{2!} + \frac{(x-3)^6}{3!} + \cdots + (-1)^{n+1}\frac{(x-3)^{2n}}{n!} + \cdots - \frac{(x-3)^{100}}{50!}.$$

What is the value of $f^{(30)}(3)$?

(A) $-\frac{30!}{15!}$ (B) $-\frac{1}{30!}$ (C) $\frac{1}{30!}$ (D) $\frac{1}{15!}$ (E) $\frac{30!}{15!}$

The nth derivative of a function f at $x = 0$ is given by $f^{(n)}(0) = (-1)^n \frac{n+1}{(n+2)2^n}$ for all $n \geq 0$. Which of the following is the Maclaurin series for f?

(A) $-\frac{1}{2} + \frac{1}{3}x - \frac{3}{32}x^2 + \frac{1}{60}x^3 - \cdots$

(B) $\frac{1}{2} - \frac{1}{3}x + \frac{3}{16}x^2 - \frac{1}{10}x^3 + \cdots$

(C) $\frac{1}{2} + \frac{1}{3}x + \frac{3}{32}x^2 + \frac{1}{60}x^3 + \cdots$

(D) $\frac{1}{2} - \frac{1}{3}x + \frac{3}{32}x^2 - \frac{1}{60}x^3 + \cdots$

(E) $\frac{1}{2} - 3x + \frac{32}{3}x^2 - 60x^3 + \cdots$

Let f be a function having derivatives of all orders for $x > 0$ such that $f(3) = 2$, $f'(3) = -1$, $f''(3) = 6$, and $f'''(3) = 12$. Which of the following is the third-degree Taylor polynomial for f about $x = 3$?

(A) $2 - x + 6x^2 + 12x^3$

(B) $2 - x + 3x^2 + 2x^3$

(C) $2 - (x-3) + 6(x-3)^2 + 12(x-3)^3$

(D) $2 - (x-3) + 3(x-3)^2 + 4(x-3)^3$

(E) $2 - (x-3) + 3(x-3)^2 + 2(x-3)^3$

AP Calculus BC – Worksheet 91 **Taylor/Maclaurin Series**

Find the Maclaurin series for the following functions.

1) e^{-5x}	2) $e^{-x/2}$	3) $5\sin(-x)$
4) $\cos\sqrt{x}$	5) xe^{x}	6) $x^2 \sin x$
7) $\frac{x^2}{2} - 1 + \cos x$	8) $\sin x - x + \frac{x^3}{3!}$	9) $x\cos \pi x$
10) $x^2 \cos\left(x^2\right)$		

Use series to evaluate each limit.

11) $\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$	12) $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$
13) $\lim_{t \to 0} \frac{1 - \cos t - \frac{t^2}{2}}{t^4}$	14) $\lim_{\theta \to 0} \frac{\sin\theta - \theta + \frac{\theta^3}{6}}{\theta^5}$

1	The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $0 < x < \frac{1}{2}$ B) converges only if $-1 < x < 1$ C) converges only if $x = 0$ D) converges for all values of x E) diverges
2	The first three terms in the Maclaurin series for $\frac{e^x + e^{-x}}{2}$ are A) $1 + \frac{x^2}{2} + \frac{x^4}{4}$ B) $1 + \frac{x^2}{2!} + \frac{x^4}{4!}$ C) $x + \frac{x^3}{3} + \frac{x^5}{5}$ D) $x + \frac{x^3}{3!} + \frac{x^5}{5!}$ E) $x + \frac{x^2}{2!} + \frac{x^3}{3!}$
3	For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n+2}$ converge? A) $-1 \le x \le 1$ B) $-3 \le x < -1$ C) $-3 \le x \le -1$ D) $-1 \le x \le 3$ E) $1 < x < 3$
4	The coefficient of $\left(x - \frac{\pi}{2}\right)^4$ in the Taylor series for $\sin x$ about $x = \frac{\pi}{2}$ is A) $-\frac{1}{2}$ B) $\frac{1}{12}$ C) $\frac{1}{24}$ D) $-\frac{1}{720}$ E) $-\frac{1}{24}$
5	The interval of convergence of the power series $\sum_{n=0}^{\infty} 5x^n$ is A) $(-\infty, \infty)$ B) $(-5, 5)$ C) $[-1, 1]$ D) $\left[-\frac{1}{5}, \frac{1}{5}\right]$ E) $(-1, 1)$
6	The coefficient of $\left(x - \frac{\pi}{2}\right)^6$ in the Taylor series for $\sin x$ about $x = \frac{\pi}{2}$ is A) $-\frac{1}{3!}$ B) -1 C) $6!$ D) $-\frac{1}{6!}$ E) $\frac{1}{3!}$
7	The coefficient of x^3 in the power series $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ is A) 1 B) $\frac{1}{6}$ C) -1 D) $\frac{1}{120}$ E) $-\frac{1}{6}$
8	If $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$ for all real x, then $f''(0) =$ A) 0 B) 1 C) $\frac{1}{2}$ D) 2 E) -2

Answers

1. D	2. B	3. B	4. C	5. E	6. D	7. E	8. E

AP Calculus BC – Worksheet 93 **Power Series – Review for the Test**

1	What is the radius of convergence for the power series $\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n$? A) $\frac{1}{3}$ B) $\frac{1}{2}$ C) 1 D) 3 E) 6
2	The first three terms in the Maclaurin Series for $\ln(1+x)$ are A) $1-\frac{x}{2}+\frac{x^2}{3}$ B) $x-\frac{x^2}{2}+\frac{x^3}{3}$ C) $x+\frac{x^2}{2}+\frac{x^3}{3}$ D) $1-\frac{x^3}{3}+\frac{x^5}{5}$ E) $1+\frac{x^3}{3}+\frac{x^5}{5}$
3	$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ is the Taylor series about zero for which of the following functions? A) $\sin x$ B) $\cos x$ C) e^x D) e^{-x} E) $\ln(1+x)$
4	The interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(5n)!}$ is A) $(-5,5]$ B) $(-\infty,\infty)$ C) 5 D) $\left(\frac{1}{5},\frac{1}{5}\right)$ E) $[-5,5]$
5	Which of the following functions is equivalent, for all values of x, to $\frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{2x^n}{n!}\right]$? A) e^{x^2} B) $2\sin x$ C) e^{2x} D) $2\cos x$ E) $2e^x$
6	If the first three terms of the Maclaurin series for e^x are $A+Bx+Cx^2$, where $A, B,$ and C are constants, then the value of $A+B+C$ is A) 3 B) 2 C) $\frac{5}{2}$ D) $\frac{5}{3}$ E) $\frac{1}{2}$
7	If $f(x)=x^7e^{x^5}$, then $f^{(17)}(0)=$ "the 17th derivative of f at 0" A) $\frac{1}{2}$ B) $\frac{17!}{6}$ C) $\frac{6}{17!}$ D) $\frac{17!}{2}$ E) 0
8	Which of the following is the Maclaurin series for $\sin x$? A) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ B) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ C) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ D) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$ E) $\sum_{n=0}^{\infty}x^n$
9	How many terms of the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$ are needed to approximate its sum with an error of less than 0.01?

Answers:

9. D	10. B	11. D	12. B	13. E	14. C	15. D	16. A	

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6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

   (a) Use the ratio test to find R.

   (b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

   (c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

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6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \ge 2$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

   (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

   (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.
